

Last time:  $O_2 \subseteq GL_2(\mathbb{R})$

$$\{g \in GL_2(\mathbb{R}) : {}^t g g = 1\}$$

$$SO_2 \subseteq O_2$$

We proved: - every elt  $p \in SO_2$  is a rotation

- Every orientation-reversing elt of  $O_2$  is a reflection about some line thru origin.

Prop<sup>n</sup> Every orientation-preserving isometry of  $\mathbb{R}^2$  is either a translation or a rotation about some pt P.

Before: If  $f \in \underbrace{M_2}_{\text{isometries of plane}}$ , then  $f = t_a p$  for some  $p \in O_2$

$$p = \begin{cases} p_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} & \text{if } \det(p) = 1 \\ p_{\theta r}, \quad r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \det(p) = -1 \end{cases}$$

$\Rightarrow$  If  $f \in M_2$  is orientation-preserving,

$$f = t_a \underbrace{p_\theta}_{\text{rotation about origin}} \text{ for some } a \in \mathbb{R}^2, \theta \in \mathbb{R}$$

Prop<sup>n</sup>  $\Rightarrow$  if  $\theta \notin 2\pi\mathbb{Z}$ , then  $t_a p_\theta$  is a rotation about some pt P in the plane.

Lemma: Suppose  $P \in \mathbb{R}^2$ , and  $\mu \in \mathbb{R}$ . Then rotation by  $\mu$  about  $P$  is:  $t_P \rho_\mu t_{-P}$

Pf Translation by  $\mu$  about  $P$ : If  $v \in \mathbb{R}^2$ , then

$$\underline{P+v \mapsto P + \rho_\mu(v)}$$

$$t_P \rho_\mu t_{-P} (P+v) = t_P \rho_\mu (v) = P + \rho_\mu(v) \quad \checkmark \quad \square$$

Pf of Prop<sup>4</sup>  $f = t_a \rho_\theta$  for some  $a \in \mathbb{R}^2$ ,  $\theta \in \mathbb{R}$ .

If  $\theta \in 2\pi\mathbb{Z}$ , then  $\rho_\theta = 1 \Rightarrow f = t_a$  is a translation.

Now suppose  $\theta \notin 2\pi\mathbb{Z}$ .

WIP:  $\underline{t_a \rho_\theta = t_P \rho_\theta t_{-P}}$  for some pt  $P$  in  $\mathbb{R}^2$

$$\begin{aligned} (\Leftarrow) \quad t_a &\stackrel{?}{=} \underline{t_P \rho_\theta t_{-P} \rho_{-\theta}} \\ &= t_P t_{-\rho_\theta(P)} \quad \underline{\text{Using: } \rho_\theta t_{-P} \rho_{-\theta} = t_{-\rho_\theta(P)} \checkmark} \\ &= \underline{t_{P - \rho_\theta(P)}} \end{aligned}$$

WIP:  $\exists P \in \mathbb{R}^2$  so that

$$a = P - \rho_\theta(P) = (I_2 - \rho_\theta)(P)$$

Note:  $1 - \rho_\theta = \begin{pmatrix} 1 - \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & 1 - \cos(\theta) \end{pmatrix}$

$$\begin{aligned} \det(1 - \rho_\theta) &= (1 - \cos(\theta))^2 + \sin(\theta)^2 \\ &= 1 - 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta) \\ &= 2 - 2\cos(\theta) \end{aligned}$$

$$-1 \leq \cos(\theta) \leq 1 \quad \text{and} \quad \theta \notin 2\pi\mathbb{Z} \Rightarrow \cos(\theta) \neq 1$$

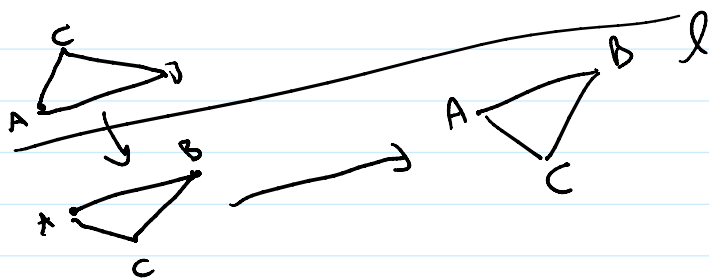
$$\Rightarrow 2 - 2\cos(\theta) \neq 0$$

$$\Rightarrow \det(1 - \rho_\theta) \neq 0$$

$$\Rightarrow P \text{ exists.} \quad \square$$

Next task: Understand the orientation-reversing isometries of the plane.

Def<sup>n</sup> A glide reflection is a reflection about some line  $l$  composed with a translation by a vector parallel to  $l$ .



A glide reflection about line  $l$  is  $t_v r_l$  where

- $r_l$  is reflection about  $l$
- $t_v$  is translation by  $v$ , with  $v$  a vector parallel to  $l$ .

Prop<sup>n</sup> Every orientation-reversing isometry  $f$  of the plane is a reflection or a glide reflection about some line  $l$  in the plane.

Pf: Every  $f$  is of form  $f = t_a \circ r$ ,  $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Two steps

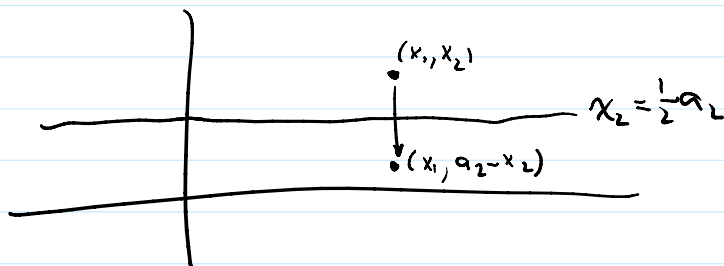
①  $f = t_a r$

② reduce to case 1.

Pf  $t_a r \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t_a \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1 + a_1 \\ a_2 - x_2 \end{pmatrix}$

where  $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Define  $l : x_2 = \frac{1}{2}a_2$ , a line



$r_l \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ a_2 - x_2 \end{pmatrix}$

$\downarrow$                        $\downarrow$                        $\downarrow$

$$l(x_2) \quad |a_2 - x_2|$$

$$t_{\underbrace{ae_1}_{\begin{pmatrix} a_1 \\ c \end{pmatrix}}} r_l \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t_{ae_1} \begin{pmatrix} x_1 \\ a_2 - x_2 \end{pmatrix} = \begin{pmatrix} x_1 + a_1 \\ a_2 - x_2 \end{pmatrix}$$

Note:  $ae_1$  is parallel to  $l$ .

$\Rightarrow t_a r = t_{ae_1} r_l$  is a glide reflection.

② General case  $f = t_a p_{e_2} r = t_a \underbrace{p_{e_2} r p_{e_2}}_{\text{reflection about } l}$



$$f = p_{e_2} (p_{e_2} t_a p_{e_2}) p_{e_2}$$

$$= p_{e_2} (t_{a'} r) p_{e_2} \quad \text{because } p_{e_2} t_a p_{e_2} = t_{a'}$$

$$\text{where } a' = p_{e_2}(a)$$

$$= p_{e_2} (t_b r_l) p_{e_2} \quad \text{where } b \text{ is a vector parallel to the line } l$$

and  $r_l$  is reflection across  $l$

$$= (p_{e_2} t_b p_{e_2}) (p_{e_2} r_l p_{e_2})$$

$$\tau_{P_{e/2}(b)} \circ P_{e/2}(l)$$

Because  $b$  is parallel to  $l$ ,  $P_{e/2}(b)$  is parallel to  $P_{e/2}(l)$   
 $\rightarrow$  is a glide reflection.  $\square$

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Next task: Understand the finite gps  $G$  that are subgps of  $O_2$

Dihedral gp  $D_n$  ( $n \geq 1$  is a pos. integer)

Denote by  $D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$

Will prove:

Thm  $D_n$  has order  $2n$ , and  $\exists$  an injective gp homom  $\varphi: D_n \rightarrow O_2$ .

Thm Suppose  $G$  is a finite subgp of  $O_2$ . Then

- ① If  $G$  is contained in  $SO_2$ ,  $G \cong C_n$ , the cyclic gp of order  $n$ .
- ② If  $G$  is not contained in  $SO_2$ ,  $G \cong D_n$ , the dihedral gp of order  $2n$ .